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Chains of Prime Ideals in Polynomial Rings*

ADA MARIA DE SOUZA DOERING AND YVES LEQUAIN

Universidade Federal do Rio Grande do Sul, Instituto de Matemática, Rua Sarmento Leite 425, 90000—Porto Alegre—Brazil, and Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina, 110, 22460—Rio de Janeiro—Brazil

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0. INTRODUCTION AND DEFINITIONS

Let R be a domain and $R[X]$ the ring of polynomials in one variable over R; let P be a prime ideal of R whose height is equal to n. Let $\mathscr{P}$ be a prime ideal of $R[X]$ such that $\mathscr{P} \cap R = P$, $\mathscr{P} \neq P[X]$. The objective of this paper is to study the saturated chains of prime ideals between (0) and $\mathscr{P}$ and between (0) and $P[X]$. By classical results of Seidenberg [11] and Jaffard [6] it is already known that $n + 1 \leqslant \text{height } \mathscr{P} \leqslant 2n + 1$ and that height $\mathscr{P} =$ height $P[X] + 1$.

In Section 2, we show that the set $\{r/r$ is the length of some saturated chain of prime ideals in $R[X]$ between (0) and $\mathscr{P}\}$ is independent of the choice of $\mathscr{P}$. In the special case of R Noetherian, this has already been proved by Houston and McAdam [5].

In Section 3, we show that for every integer t such that $n + 1 \leqslant t \leqslant$ height $\mathscr{P}$, there exists a saturated chain of prime ideals in $R[X]$ between (0) and $\mathscr{P}$ whose length is equal to t; similarly, we show that for every integer u such that $n + 1 \leqslant u \leqslant$ height $P[X]$, there exists a saturated chain of prime ideals in $R[X]$ between (0) and $P[X]$ whose length is equal to u. Furthermore, we show that such chains can be chosen such that the chains of the intersections with R are also saturated.

In Section 4, we show that there is no rule at all that governs the existence of saturated chain of prime ideals in $R[X]$ between (0) and $\mathscr{P}$ whose length are less than or equal to n. More precisely, we show that given two positive integers n and m such that $n + 1 \leqslant m \leqslant 2n + 1$, and given integers $u_1, \ldots, u_s$ such that $2 \leqslant u_1 < \cdots < u_s \leqslant n$, there exists a domain R and a prime ideal P of R with height equal to n such that, for every prime ideal $\mathscr{P}$ of $R[X]$ such

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that $\mathscr{P} \cap R = P$ and $\mathscr{P} \neq P[X]$, we have $\{u_1,..., u_s, n+1, n+2,..., m\} = \{r/r$ is the length of some saturated chain of prime ideals in $R[X]$ between (0) and $\mathscr{P}\}$. We observe that, whereas in general there is no such domain that be integrally closed, there always exists one that is seminormal. If $m = n + 1$, we show that there is such a domain that is Noetherian.

An important tool in that study is given by the following Principal Ideal Theorem for (not necessarily Noetherian) polynomial rings that we prove in Section 1. If $f(X) \in \mathscr{P}\backslash P[X]$, then (i) there exists a prime ideal $\mathscr{Q}$ of $R[X]$ such that $\mathscr{Q} \cap R = (0)$, $f(X) \in \mathscr{Q}$ and $\mathscr{Q} \subseteq \mathscr{P}$, and (ii) height $(\mathscr{P}/(f(X))) =$ height P. Exhibiting examples, we note that this is essentially the best possible result.

In this paper all rings are commutative with identity, and a prime ideal is always different from the unit ideal. The symbol $\subseteq$ denotes inclusion and the symbol $\subset$ denotes proper inclusion. Two prime ideals P and Q are *consecutive*, or *adjacent*, if $P \subset Q$ and $\text{height}(Q/P) = 1$. A chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_r$ is *saturated* if $\text{height}(P_i/P_{i-1}) = 1$ for every $i = 1,..., r$; in this case r is the *length* of the chain. If R is a ring, a prime ideal $\mathscr{P}$ of $R[X]$ such that $\mathscr{P} \neq (\mathscr{P} \cap R)[X]$ is an *upper* or, more precisely, an *upper to* $\mathscr{P} \cap R$. If $\mathscr{P}_0 \subset \mathscr{P}_1 \subset \cdots \subset \mathscr{P}_r$ is a chain of prime ideals of $R[X]$, the set $\{\mathscr{P}_i \cap R \mid i = 0,..., r\}$ is the *set of intersections with* R of the chain; with the order of inclusion, that set is the *chain of intersections with* R of the chain. A domain D is *seminormal* if a belongs to D whenever a is an element of the quotient field of D such that a^2 and a^3 belong to D.

1. Principal Ideal Theorem for Polynomial Rings

In this section we prove a theorem of the type "Krull's Principal Ideal Theorem" for (not necessarily Noetherian) polynomial rings. It will be central in the study of the saturated chain of prime ideals of a polynomial ring that we will do in the subsequent sections.

THEOREM A. *Let R be a domain, P a prime ideal of R and $\mathscr{P}$ an upper to P. Let $f(X) \in \mathscr{P}\backslash P[X]$. Then there exists an upper to zero $\mathscr{Q}$ such that $f(X) \in \mathscr{Q} \subseteq \mathscr{P}$.*

We first prove the following lemma:

LEMMA 1.1. *Let R be a domain and let $I \neq (0)$ be an ideal of $R[X]$. Then,*

(a) *There exists an upper to zero $\mathscr{Q}$ such that $I \subseteq \mathscr{Q}$ if and only if $I \cap R = (0)$.*

(b) *If $\mathscr{P}$ is a prime ideal of $R[X]$ that contains I, then there exists an upper to zero $\mathscr{Q}$ such that $I \subseteq \mathscr{Q} \subseteq \mathscr{P}$ if and only if $IR[X]_{\mathscr{P}} \cap R = (0)$.*

(c) *For every prime ideal $\mathscr{P}$ of $R[X]$ that contains I, there exists an upper to zero $\mathscr{Q}$ such that $I \subseteq \mathscr{Q} \subseteq \mathscr{P}$ if and only if $R[X]/\sqrt{I}$ is R-torsion free.*

Proof. (a) The necessity is clear. For the sufficiency, consider the multiplicative system $S = R\backslash\{0\}$ and take $\mathscr{Q}$ to be an ideal of $R[X]$ that contains I and that is maximal for the property to have empty intersection with S.

(b) Let $\mathscr{P}$ be a prime ideal of $R[X]$ that contains I. If $\mathscr{Q}$ is an upper to zero such that $I \subseteq \mathscr{Q} \subseteq \mathscr{P}$, then we have $IR[X]_{\mathscr{P}} \cap R \subseteq \mathscr{Q}R[X]_{\mathscr{P}} \cap R \subseteq \mathscr{Q}R[X]_{\mathscr{P}} \cap R[X] \cap R = \mathscr{Q} \cap R = (0)$. Conversely, if $IR[X]_{\mathscr{P}} \cap R = (0)$, consider the multiplicative system $S = R\backslash\{0\}$ and take $\mathscr{Q}$ to be an ideal of $R[X]$ that contains I and that is maximal for the double property to be contained in $\mathscr{P}$ and to have empty intersection with S.

(c) Suppose that there exists $0 \neq r \in R$ and $f(X) \in R[X]\backslash\sqrt{I}$ such that $rf(X) \in \sqrt{I}$, consider the multiplicative system $S = \{1, f(X), ..., f^n(x), ...\}$ and take $\mathscr{P}$ to be an ideal of $R[X]$ that contains $\sqrt{I}$ and is maximal for the property to have empty intersection with S. Since $rf(X) \in \sqrt{I}$, there exists an integer n such that $r^n(f(X))^n \in I$; since $(f(X))^n \notin \mathscr{P}$, we obtain that $0 \neq r^n = (r^n(f(X))^n)/(f(X))^n \in IR[X]_{\mathscr{P}} \cap R$; by (b), this implies that there is no upper to zero $\mathscr{Q}$ such that $I \subseteq \mathscr{Q} \subseteq \mathscr{P}$. Conversely, suppose that $R[X]/\sqrt{I}$ is R-torsion free and let $\mathscr{P}$ be a prime ideal of $R[X]$ that contains I. Let $r \in IR[X]_{\mathscr{P}} \cap R$; then there exists $f(X) \in R[X]\backslash\mathscr{P}$ such that $rf(X) \in I$; since $R[X]/\sqrt{I}$ is R-torsion free, we obtain that $r = 0$; by (b), this implies that there exists an upper to zero $\mathscr{Q}$ such that $I \subseteq \mathscr{Q} \subseteq \mathscr{P}$.

Proof of Theorem A. By localizing at P, we can suppose without loss of generality that P is the only maximal ideal of R. In view of Lemma 1.1(c), it suffices to show that $R[X]/\sqrt{(f(X))}$ is R-torsion free; since R is a domain, it even suffices to show that $R[X]/(f(X))$ is R-torsion free. Let $g(X) \in R[X]$ and $0 \neq r \in R$ such that $rg(X) \in (f(X))$, say, $rg(X) = f(X)\,h(X)$ with $h(X) \in R[X]$. We want to show that $g(X) \in (f(X))$. Looking at the contents of these polynomials, there exists an integer n such that $c(fh)(c(f))^n = c(h)(c(f))^{n+1}$ [9, Lemma 1, p. 283]; since $f(X) \notin P[X]$, we have $c(f) = R$ and consequently $c(fh) = c(h)$; then $rc(g) = c(h)$ and the coefficients of $h(X)$ are multiples of r, i.e., there exists $h_1(X) \in R[X]$ such that $h(X) = rh_1(X)$. Then, we obtain $rg(X) = f(X)\,h(X) = rf(X)\,h_1(X)$, hence $g(X) = f(X)\,h_1(X) \in (f(X))$ since r is not a zero divisor.

COROLLARY A.1. *Let R be a ring and $P_0 \subset \cdots \subset P_r$ a saturated chain of prime ideals of R. Let $\mathscr{P}_r$ be an upper to P_r.*

(a) *If $f(X) \in \mathscr{P}_r \backslash P_r[X]$, then for $i = 0,\ldots, r-1$ there exists $\mathscr{P}_i$, upper to P_i, such that $\mathscr{P}_0 \subset \cdots \subset \mathscr{P}_r$ is saturated and $f(X) \in \mathscr{P}_0$.*

(b) *If the chain of extended primes $P_0[X] \subset \cdots \subset P_r[X]$ is not saturated, then there exists a saturated chain of prime ideals of length $(r+1)$ between $P_0[X]$ and $P_r[X]$.*

Proof. (a) Applying Theorem A to the domain R/P_{r-1}, we obtain $\mathscr{P}_{r-1}$, upper to P_{r-1} such that $f(X) \in \mathscr{P}_{r-1} \subset \mathscr{P}_r$. Now, if $\mathscr{Q}$ is a prime ideal of $R[X]$ such that $\mathscr{P}_{r-1} \subset \mathscr{Q} \subseteq \mathscr{P}_r$, then we must have $\mathscr{Q} \cap R = \mathscr{P}_r \cap R = P_r$; this implies that $\mathscr{Q} = \mathscr{P}_r$ for otherwise we would have $f(X) \in \mathscr{P}_{r-1} \subset \mathscr{Q} = P_r[X]$ which would contradict the fact that $f(X) \notin P_r[X]$; thus $\mathscr{P}_{r-1} \subset \mathscr{P}_r$ is saturated. Now we can proceed by induction since $f(X) \in \mathscr{P}_{r-1} \backslash P_{r-1}[X]$.

(b) If the chain $P_0[X] \subset \cdots \subset P_r[X]$ is not saturated, let $k = \sup\{j \mid P_j[X] \subset P_{j+1}[X] \text{ is not saturated}\}$. Since $P_k[X] \subset P_{k+1}[X]$ is not saturated, there exists $\mathscr{P}_k$, upper to P_k, such that $\mathscr{P}_k \subset P_{k+1}[X]$. By (a), for every $i = 0,\ldots, k-1$, there exists $\mathscr{P}_i$, upper to P_i, such that $\mathscr{P}_0 \subset \cdots \subset \mathscr{P}_k$ is saturated; then the chain $P_0[X] \subset \mathscr{P}_0 \subset \cdots \subset \mathscr{P}_k \subset P_{k+1}[X] \subset \cdots \subset P_r[X]$ is saturated with length equal to $r+1$.

We can give another version of Theorem A:

THEOREM A′. *Let R be a ring, P a prime ideal of R and $\mathscr{P}$ an upper to P. Let $f(X) \in \mathscr{P} \backslash P[X]$. Then* $\operatorname{height}(\mathscr{P}/(f(X))) = \operatorname{height} P$.

Proof. By Corollary A.1(a), we have $\operatorname{height}(\mathscr{P}/(f(X))) \geqslant \operatorname{height} P$. On the other hand, if $\mathscr{P}_0 \subset \cdots \subset \mathscr{P}_t = \mathscr{P}$ is a chain of prime ideals in $R[X]$ such that $f(X) \in \mathscr{P}_0$, then every $\mathscr{P}_i$ must be an upper to $\mathscr{P}_i \cap R$ since $f(X) \notin P[X]$; in particular we must have $\mathscr{P}_i \cap R \neq \mathscr{P}_j \cap R$ for every $i \neq j$, and consequently $\operatorname{height}(\mathscr{P}/(f(X))) \leqslant \operatorname{height} P$.

COROLLARY A.2. *Let R be a ring, $Q \subset P$ prime ideals of R and $\mathscr{P}$ an upper to P. Let $\mathscr{F} = \{$Uppers to Q that are contained in $\mathscr{P}\}$. Then $\mathscr{P} = P[X] \cup (\bigcup_{\mathscr{Q} \in \mathscr{F}} \mathscr{Q})$ and $\mathscr{F}$ is infinite.*

Remarks. (1) The result given in Theorem A (or equivalently in Theorem A′) is the best possible in as much as the hypothesis "$f(X) \notin P[X]$" cannot be avoided, even if we accept to weaken the conclusion to "there exists a height-one prime ideal $\mathscr{Q}$ such that $f(X) \in \mathscr{Q} \subseteq \mathscr{P}$." As a matter of fact, in general, a nonrestrictive Principal Ideal Theorem will not be valid on $R[X]$ even if such a nonrestrictive theorem is valid on R. Indeed, let (R, P) be a one-dimensional quasi-local domain such that the dimension of $R[X]$ is 3, let $0 \neq a \in P$ and consider $\mathscr{P} = (P, X)$. Since the height of $P[X]$ is 2, the only height-one prime ideals of $R[X]$ are uppers to (0) and, clearly, none of them contains the constant polynomial a. Observe that if we take a polynomial $f(X) \in P[X]$ of degree

$\geqslant 1$, then by Lemma 1.1(a), such a polynomial will be contained in an upper to (0), but in general will not be contained in any height-one prime ideal contained in $(P. X)$; this is the case for example with $f(X) = a + aX = a(1+X)$ since $1 + X \notin (P, X)$ and since a belongs to no height-one prime ideal of $R[X]$. This shows that the result of Corollary A.2 could not be strengthened to $\mathscr{P} = P \cup (\bigcup_{\mathscr{Q} \in \mathscr{F}} \mathscr{Q})$.

(2) When R is a Noetherian ring and $\mathscr{P}$ is an upper, we have $\text{height}(\mathscr{P} \cap R) = (\text{height}\, \mathscr{P}) - 1$; thus in this case, our Theorem A′ takes the usual form $\text{height}(\mathscr{P}/(f(X))) = (\text{height}\, \mathscr{P}) - 1$.

(3) If $f(X)$ is a polynomial of degree d, then $f(X)$ is contained in at most d uppers to zero [7, Theorem 36, p. 25].

2. Saturated Chains in Polynomial Rings

In this section we generalize some results that have been proved in the Noetherian case by Houston and McAdam [5]. The main one is the following:

Theorem B. *Let R be a domain and P a prime ideal of R. Let $\mathscr{P}$ be ab upper to P. Then $\{r \mid r$ is the length of some saturated chain of prime ideals in $R[X]$ between (0) and $\mathscr{P}\}$ is independent of the choice of $\mathscr{P}$.*

Lemma 2.1 [5, Proposition 1.1, p. 742]. *Let R be a domain that is integrally closed and let $\mathscr{Q}$ be an upper to zero. Then $\mathscr{Q}$ is generated by its polynomials of minimal degree.*

Proposition 2.2. *Let R be a domain and P a prime ideal of R. Let $\mathscr{P}$ be an upper to P and $\mathscr{Q}$ an upper to (0) such that $\mathscr{Q} \subset \mathscr{P}$ and $\text{height}(\mathscr{P}/\mathscr{Q}) < \text{height}\, P$. Then $P[X]$ contains the polynomials of minimal degree in $\mathscr{Q}$.*

Proof. Let $g(X)$ be a polynomial of minimal degree in $\mathscr{Q}$; $\mathscr{Q}$ is the only upper to zero that contains $g(X)$ [7, Theorem 36, p. 25]. Suppose that $g(X) \notin P[X]$ and let $r = \text{height}\, P$. By Corollary A.1, there exists a saturated chain of prime ideals of length r, say $\mathscr{P}_0 \subset \cdots \subset \mathscr{P}_r = \mathscr{P}$, with $\mathscr{P}_0$ an upper to zero that contains $g(X)$; since $\mathscr{Q}$ is the only upper to zero that contains $g(X)$, one must have $\mathscr{P}_0 = \mathscr{Q}$ and, consequently, $\text{height}(\mathscr{P}/\mathscr{Q}) \geqslant r$; this contradicts the hypothesis.

Theorem 2.3. *Let R be a domain with integral closure R'. Let P be a prime ideal of R and $\mathscr{P}$ an upper to P. Let s be an integer. Then, the following statements are equivalent:*

(i) *There exists an upper to zero* $\mathscr{Q}$ *such that* $\mathscr{Q} \subseteq \mathscr{P}$, $\mathscr{Q} \not\subseteq P[X]$, $\operatorname{height}(\mathscr{P}/\mathscr{Q}) = s$.

(ii) *There exists a finite R-algebra* R^* *such that* $R \subseteq R^* \subseteq R'$ *that possesses a prime ideal of height s lying over P.*

Notice that (ii) is independent of the choice of $\mathscr{P}$.

Proof. (i) $\Rightarrow$ (ii). Since $\mathscr{Q} \subseteq P[X]$, every prime ideal between $\mathscr{Q}'$ and $\mathscr{P}$ is an upper and therefore $\operatorname{height}(\mathscr{P}/\mathscr{Q}) \leqslant \operatorname{height} P$. If $\operatorname{height}(\mathscr{P}/\mathscr{Q}) = \operatorname{height} P$, then we can take $R^* = R$. If $t = \operatorname{height}(\mathscr{P}/\mathscr{Q}) < \operatorname{height} P$, let $\mathscr{Q} = \mathscr{Q}_0 \subset \cdots \subset \mathscr{Q}_t = \mathscr{P}$ be a saturated chain of prime ideals in $R[X]$ of length t between $\mathscr{Q}$ and $\mathscr{P}$. By the Going-up Theorem there exists a saturated chain of prime ideals in $R'[X]$, $\mathscr{Q}'_0 \subset \mathscr{Q}'_1 \subset \cdots \subset \mathscr{Q}'_t$ with $\mathscr{Q}'_i \cap R[X] = \mathscr{Q}_i$. Let $P' = \mathscr{Q}'_t \cap R'$; since $R'[X]$ is integral over $R[X]$, we have that $\mathscr{Q}'_t$ is an upper to P' and that $\mathscr{Q}'_0$ is an upper to zero. Since R' is integrally cosed, $\mathscr{Q}'_0$ is generated by its polynomials of minimal degree by Lemma 2.1; since $\mathscr{Q}_0 = \mathscr{Q} \not\subseteq P[X]$, we also have $\mathscr{Q}'_0 \not\subseteq P'[X]$; then there exists a polynomial of minimal degree in $\mathscr{Q}'_0$, say $f(X)$, such that $f(X) \notin P'[X]$. Let $d_0, d_1, ..., d_t$ be the coefficients of $f(X)$ and let $R^* = R[d_0, d_1, ..., d_t]$. Let $\mathscr{Q}^*_i = \mathscr{Q}'_i \cap R^*[X]$ for $i = 0, ..., t$ and let $P^* = P' \cap R^*$; clearly P^* lies over P. Since $R^*[X]$ is integral over $R[X]$, we have that $\mathscr{Q}^*_t$ is an upper to P^*, that $\mathscr{Q}^*_0$ is an upper to zero and that $\operatorname{height}(\mathscr{Q}^*_t/\mathscr{Q}^*_0) = t$. Furthermore, $f(X)$ is certainly a polynomial of minimal degree in $\mathscr{Q}^*_0$ and $f(X) \notin P^*[X]$; then by Proposition 2.2, we have $t = \operatorname{height} P^*$.

(ii) $\Rightarrow$ (i). Let R^* be a finite R-algebra such that $R \subseteq R^* \subseteq R'$ possesses a prime ideal P^* of height s lying over P. We claim that since R^* is a finite R-module, there exists only a finite number of prime ideals of R^* that lie over P. Indeed, it clearly suffices to show that if d is an integral element over a ring A and if $\mathfrak{A}$ is a prime ideal of A, then there exists only a finite number of prime ideals of $A[d]$ that lie over $\mathfrak{A}$. For this, consider the natural surjection $\varphi: A[Y] \to A[d]$; the prime ideals of $A[d]$ that lie over $\mathfrak{A}$ correspond to the prime ideals of $A[Y]$ that contain $\ker \varphi$ and whose intersection with A is equal to $\mathfrak{A}$; since $\ker \varphi$ contains a monic polynomial, the intersection of those prime ideals is not equal to $\mathfrak{A}[Y]$ and there can be only a finite number of them [7, Theorem 36, p. 25]. The claim being proved, let $P^*_1 = P^*, ..., P^*_t$ be the prime ideals of R^* that lie over P. The prime ideals of $R^*[X]$ that lie over $\mathscr{P}$ are finite in number [8, Theorem 2, p. 707] and incomparable between themselves; everyone of them is an upper to P^*_i for some $i \in \{1, ..., t\}$. By the Going-up Theorem, there does exist some upper to P^*, say $\mathscr{P}^*$, that lies over $\mathscr{P}$. Let $f(X) \in \mathscr{P}^*$ such that $f(X)$ belongs to no other prime ideal of $R^*[X]$ that lies over $\mathscr{P}$ and such that $f(X) \notin P^*[X]$. Since height $P^* = s$, there exists by Corollary A.1, a saturated chain of prime ideals in $R^*[X]$ of length s, $\mathscr{Q}^*_0 \subset \cdots \subset \mathscr{Q}^*_1 = \mathscr{P}^*$ wih $\mathscr{Q}^*_0$ an upper to zero that contains $f(X)$; let $\mathscr{Q} = \mathscr{Q}_0^* \cap R[X]$. It is clear that $\mathscr{Q}$ is an upper to zero that is contained in $\mathscr{P}$; furthermore $\mathscr{Q}$ is not contained in $P[X]$ because if it were then by the Going-up Theorem, $\mathscr{Q}_0^*$ would have to be contained in $P_i^*[X]$ for some $i \in \{1,..., t\}$, which is absurd since $f(X) \in \mathscr{Q}_0^*$ and $f(X) \notin P_i^*[X]$ for every $i \in \{1,..., t\}$. Finally height$(\mathscr{P}/\mathscr{Q}) = s$; indeed, suppose that height$(\mathscr{P}/\mathscr{Q}) > s$; then by the Going-up Theorem, there exists a saturated chain of prime ideals in $R^*[X]$ of length strictly bigger than s between $\mathscr{Q}_0^*$ and some prime ideal lying over $\mathscr{P}$; the latter has to be $\mathscr{P}^*$ since $f(X)$ belongs to it; this implies that height$(\mathscr{P}^*/\mathscr{Q}_0^*) > s = \text{height } P^*$, which is absurd since every prime ideal between $\mathscr{Q}_0^*$ and $\mathscr{P}^*$ is necessarily an upper

Proof of Theorem B. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two uppers to P. We shall prove that if r is the length of some saturated chain of primes between (0) and $\mathscr{P}_1$, then r is also the length of some saturated chain of primes between (0) and $\mathscr{P}_2$. Let $(0) \subset \mathscr{Q}_1 \subset \cdots \subset \mathscr{Q}_{r-1} \subset \mathscr{P}_1$ be a saturated chain of prime ideals. If $\mathscr{Q}_{r-1} = P[X]$, then the chain $(0) \subset \mathscr{Q}_1 \subset \cdots \subset \mathscr{Q}_{r-1} \subset \mathscr{P}_2$ is also saturated and we are through. If $\mathscr{Q}_{r-1} \neq P[X]$, then $\mathscr{Q}_{r-1}$ is clearly an upper to some prime ideal M of R with $M \subset P$. Now we do an induction on r. If $r = 1$, the result is trivial. If $r = 2$, $\mathscr{Q}_{r-1}$ is necessarily an upper to zero and the result is a consequence of Theorem 2.3 with $s = 1$. Now, let $r \geqslant 3$ and suppose the result to be true for $r - 1$. In $(R/M)[X] \simeq (R[X]/M[X])$ we have the saturated chain $(0) \subset (\mathscr{Q}_{r-1}/M[X]) \subset (\mathscr{P}_1/M[X])$; observe that $\mathscr{Q}_{r-1}/M[X]$ is an upper to zero and that $\mathscr{P}_1/M[X]$ is an upper to P/M. Then, by the case $r = 2$, there exists a saturated chain in $(R/M)[X]$ of length equal to 2 between (0) and $\mathscr{P}_2/M[X]$ that can be lifted to a saturated chain $M[X] \subset \mathscr{M} \subset \mathscr{P}_2$ in $R[X]$ where $\mathscr{M}$ is an upper to M. Both $\mathscr{Q}_{r-1}$ and $\mathscr{M}$ are uppers to M, and we are given a saturated chain of prime ideals of length $r - 1$ between (0) and $\mathscr{Q}_{r-1}$; then, by the hypothesis of induction, there exists a saturated chain of length $r - 1$ between (0) and $\mathscr{M}$ and, therefore, a saturated chain of length r between (0) and $\mathscr{P}_2$.

COROLLARY 2.4. *Let R be a Prüfer domain of dimension n. Then $R[X]$ is catenarian of dimension $n + 1$.*

Proof. Since $R[X]$ will be catenarian if $R_M[X]$ is catenarian for every maximal ideal M of R, we can suppose that R is a valuation ring. Let Q be a prime ideal of R and $\mathscr{Q}$ an upper to Q. Let $(0) = \mathscr{P}_0 \subset \mathscr{P}_1 \subset \cdots \subset \mathscr{P}_s = \mathscr{Q}$ be a saturated chain of prime ideals between (0) and $\mathscr{Q}$; let $i = \inf\{j/\mathscr{P}_j$ is an upper to $\mathscr{P}_j \cap R\}$; let $P = \mathscr{P}_i \cap R$. We clearly have $\mathscr{P}_{i-1} = P[X]$; hence also $\mathscr{P}_e = (\mathscr{P}_e \cap R)[X]$ for every $e \leqslant i - 1$; then we have $i - 1 = \text{height } P$. Now let $j \geqslant 1$; $\mathscr{P}_j$ is an upper to $\mathscr{P}_j \cap R$ since $\mathscr{P}_i$ is an upper to $\mathscr{P}_i \cap R$ [3, (16.10), p. 218]; by Theorem 2.3, $\mathscr{P}_{j-1} \cap R \subset \mathscr{P}_j \cap R$ are consecutive since

$R/(\mathscr{P}_{j-1} \cap R)$ is integrally closed. Then we obtain $s - i = \text{height}(Q/P)$, and $s = \text{height}(Q/P) + (\text{height } P) + 1 = (\text{height } Q) + 1$.

3. Long Saturated Chains in Polynomial Rings

Given a domain R and a prime ideal $\mathscr{Q}$ of $R[X]$, we study in this section the saturated chains of prime ideals between (0) and $\mathscr{Q}$ whose length is bigger than height$(\mathscr{Q} \cap R)$.

THEOREM C. *Let R be a domain, P a prime ideal of R and $\mathscr{P}$ an upper to P. Let n be the height of P and let $m - 1$ be the height of $P[X]$. Then*:

(a) height $\mathscr{P} = m$ *and* $n + 1 \leqslant m \leqslant 2n + 1$.

(b) *For every t such that $n + 1 \leqslant t \leqslant m$, there exists a saturated chain of prime ideals of length t between* (0) *and $\mathscr{P}$ whose chain of intersections with R is saturated.*

(c) *For every u such that $n + 1 \leqslant u \leqslant m - 1$, there exists a saturated chain of prime ideals of length u between* (0) *and $P[X]$ whose chain of intersections with R is saturated.*

(d) *There may exist, or not, a saturated chain of prime ideals of length n between* (0) *and $P[X]$.*

The results contained in part (a) are well-known: the first one is an immediate consequence of Jaffard's Special Chain Theorem [6, Théorème 3, p. 3], or can be found explicitely proved in a direct way in [1, Lemma 1, p. 28]; the second one is a classical result of Seidenberg [11, Theorem 2, p. 506]. To see the results of part (d): if R is a Noetherian domain, then there exists a saturated chain of prime ideals of length n between (0) and $P[X]$; if (R, P) is a one-dimensional quasi-local integrally closed domain that is not a valuation ring, then there is no saturated chain of prime ideals of length 1 between (0) and $P[X]$ since height $P[X] = 2$ [11, Theorem 8, p. 511]. The proof of the other results of Theorem C will rely on the following result:

THEOREM C′. *Let R be a domain, P a prime ideal of R of height n, $\mathscr{P}$ an upper to P and t an integer $\geqslant n + 2$. Let $(0) = \mathscr{Q}_0 \subset \mathscr{Q}_1 \subset \cdots \subset \mathscr{Q}_t = \mathscr{P}$ be a saturated chain of prime ideals and let $\mathscr{S} = \{\mathscr{Q}_i \cap R / i = 0,..., t\}$ be its set of intersections with R. Then, there exists a saturated chain of prime ideals of length $t - 1$ between* (0) *and $\mathscr{P}$ that admits $\mathscr{S}$ as set of intersections with R.*

Proof. First we claim that there exists a saturated chain of prime ideals of $R[X]$ $(0)=\mathscr{P}_0\subset\mathscr{P}_1\subset\cdots\subset\mathscr{P}_t=\mathscr{P}$ that admits $\mathscr{S}$ as set of intersections with R such that $\mathscr{P}_i\neq(\mathscr{P}_i\cap R)[X]$ or $\mathscr{P}_{i+1}\neq(\mathscr{P}_{i+1}\cap R)[X]$ for every $i=0,\dots,t-1$. If our given chain of prime ideals is not of that type, let $k=\sup\{j/\mathscr{Q}_j=(\mathscr{Q}_j\cap R)[X]$ and $\mathscr{Q}_{j+1}=(\mathscr{Q}_{j+1}\cap R)[X]\}$; let $Q_k=\mathscr{Q}_k\cap R$ and $Q_{k+1}=\mathscr{Q}_{k+1}\cap R$. Notice that $k\leqslant t-2$, that $\mathscr{Q}_{k+2}$ is an upper to Q_{k+1} and that $\{Q_k,Q_{k+1}\}$ is the set of intersections with R of the sequence $\mathscr{Q}_k\subset\mathscr{Q}_{k+1}\subset\mathscr{Q}_{k+2}$. By Corollary A.1, there exists $\mathscr{P}'$, upper to Q_k, such that $\mathscr{P}'\subset\mathscr{Q}_{k+2}$ is saturated; then the sequance $\mathscr{Q}_k=Q_k[X]\subset\mathscr{P}'\subset\mathscr{Q}_{k+2}$ is saturated and admits $\{Q_k,Q_{k+1}\}$ as set of intersections with R. Taking $\mathscr{P}'_{k+1}=\mathscr{P}'$ and $\mathscr{P}'_i=\mathscr{Q}_i$ for every $i\neq k+1$, we get a saturated chain of prime ideals $(0)=\mathscr{P}'_0\subset\cdots\subset\mathscr{P}'_t=\mathscr{P}$ that admits $\mathscr{S}$ as set of intersections with R such that $\sup\{j/\mathscr{P}'_j=(\mathscr{P}'_j\cap R)[X]$ and $\mathscr{P}'_{j+1}=(\mathscr{P}'_{j+1}\cap R)[X]\}<k$. Then, by an induction on k we shall obtain a saturated chain of prime ideals $(0)=\mathscr{P}_0\subset\cdots\subset\mathscr{P}_t=\mathscr{P}$ as desired.

Let $i(0)=0<i(1)<\cdots<i(k)$ be the integers such that $\mathscr{P}_{i(0)}=(0)$, $\mathscr{P}_{i(1)},\dots,\mathscr{P}_{i(k)}$ are the elements of the chain such that $\mathscr{P}_{i(j)}=(\mathscr{P}_{i(j)}\cap R)[X]$. Notice that $k\geqslant 1$ since $t\geqslant n+2$; notice also that for any $j\in\{0,1,\dots,k-1\}$ we have: $\mathscr{P}_{i(j)+1}\cap R$, $\mathscr{P}_{i(j)+2}\cap R,\dots,\mathscr{P}_{i(j+1)}\cap R$ are distinct prime ideals of R. We claim that there exists $j\in\{0,1,\dots,k-1\}$ such that the chain of prime ideals of R, $\mathscr{P}_{i(j)+1}\cap R\subset\cdots\subset\mathscr{P}_{i(j+1)}\cap R$, is saturated. Indeed, otherwise we would have $\text{height}((\mathscr{P}_{i(j+1)}\cap R)/(\mathscr{P}_{i(j)}\cap R))\geqslant i(j+1)-i(j)$ for every $j=0,1,\dots,k-1$; hence also

$$\begin{aligned}
n&=\text{height}(P)\\
&\geqslant\text{height}(P/(\mathscr{P}_{i(k)}\cap R))\\
&\quad+\sum_{j=0}^{k-1}\text{height}((\mathscr{P}_{i(j+1)}\cap R)/(\mathscr{P}_{i(j)}\cap R))\\
&\geqslant(t-i(k)-1)+\sum_{j=0}^{k-1}(i(j+1)-i(j))=t-1>t-2
\end{aligned}$$

which contradicts the hypothesis on n.

Then, let j be such that the chain $\mathscr{P}_{i(j)+1}\cap R\subset\cdots\subset\mathscr{P}_{i(j+1)}\cap R$ is saturated; it has length equal to $i(j+1)-i(j)-1$. Then, since $\mathscr{P}_{i(j+1)+1}$ is an upper of $\mathscr{P}_{i(j+1)}\cap R$, there exists by Corollary A.1(a) a saturated chain of prime ideals of length $i(j+1)-i(j)$ between $(\mathscr{P}_{i(j)+1}\cap R)[X]=\mathscr{P}_{i(j)}$ and $\mathscr{P}_{i(j+1)+1}$ admitting $\{\mathscr{P}_{i(j)+1}\cap R,\dots,\mathscr{P}_{i(j+1)}\cap R\}$ as set of intersections with R. This proves our theorem since the original chain between $\mathscr{P}_{i(j)}$ and $\mathscr{P}_{i(j+1)+1}$, namely, $\mathscr{P}_{i(j)}\subset\mathscr{P}_{i(j)+1}\subset\cdots\subset\mathscr{P}_{i(j+1)}\subset\mathscr{P}_{i(j+1)+1}$, had length $i(j+1)-i(j)+1$ and had $\{\mathscr{P}_{i(j)+1}\cap R,\dots,\mathscr{P}_{i(j+1)}\cap R\}$ as set of intersections with R.

Proof of Therem C. (a) and (d) have already been checked.

(b) Since height $\mathscr{P} = m$, there exists a saturated chain of prime ideals $(0) = \mathscr{P}_0 \subset \mathscr{P}_1 \subset \cdots \subset \mathscr{P}_m = \mathscr{P}$ between (0) and $\mathscr{P}$ such that $(\mathscr{P}_i \cap R)[X]$ is a member of the chain for every $i = 0,..., m$ [6, Théorème 3, p. 35]; it is immediate to check that there even exists such a chain with the additional property that its chain of intersections is saturated. Then applying Theorem C′ step by step, we shall get, for every t such that $n + 1 \leqslant t \leqslant m$, a saturated chain of prime ideals of length t between (0) and $\mathscr{P}$ whose chain of intersections with R is the same as the preceeding one, hence in particular is saturated.

(c) Since height $(P, X) = m$, we easily get, again by Jaffard's theorem [6, Théorème 3, p. 35], that there exists a saturated chain of prime ideals of length $(m - 1)$ between (0) and $P[X]$ whose chain of intersections with R is saturated. By induction suppose that u is an integer such that $n + 1 \leqslant u \leqslant m - 1$ and suppose that there exists a saturated chain of prime ideals $(0) = \mathscr{P}_0 \subset \mathscr{P}_1 \subset \cdots \subset \mathscr{P}_{u+1} = P[X]$ of length $(u + 1)$ between (0) and $P[X]$ whose chain of intersections with R is saturated. Since $u + 1 \geqslant n + 2 = (\text{height}\, P) + 2$, we can consider $j = \sup\{i/\mathscr{P}_i \neq (\mathscr{P}_i \cap R)[X]\}$. Let $P_j = \mathscr{P}_j \cap R$; we have height $P_j \leqslant (\text{height}\, P) - (u + 1 - j) = n + j - (u + 1)$; since furthermore $u + 1 \geqslant n + 2$, we obtain that $j \geqslant (\text{height}\, P_j) + 2$. By Theorem C′, there exists a saturated chain of prime ideals $(0) = \mathscr{Q}_0 \subset \mathscr{Q}_1 \subset \cdots \subset \mathscr{Q}_{j-1} = \mathscr{P}_j$ of length $(j - 1)$ between (0) and $\mathscr{P}_j$ whose set of intersections with R is $\{\mathscr{P}_i \cap R/i = 0,..., j\}$. Then, it is clear that the chain $(0) = \mathscr{Q}_0 \subset \mathscr{Q}_1 \subset \cdots \subset \mathscr{Q}_{j-1} = \mathscr{P}_j \subset \mathscr{P}_{j+1} \subset \cdots \subset \mathscr{P}_{u+1} = P[X]$ is a saturated chain of length u between (0) and $P[X]$ whose chain of intersections with R is saturated.

Remark. Let us keep the notations of Theorem C. For $t = n + 1$, the following stronger result is true by our Corollary A.1: there exists a saturated chain of prime ideals of length t between (0) and $\mathscr{P}$ whose chain of intersections with R has length equal to height P. For $t \geqslant n + 2$, that stronger result is not valid anymore. Indeed, it is easy to check that the domain R with maximal ideal M constructed in [1, Example 4, p. 42] is such that height $M = 3$, height$(M, X) = 5$ and all the saturated chain of prime ideals of length 5 between (0) and (M, X) have a chain of intersections with R of length equal to 2.

4. Short Saturated Chains in Polynomial Rings

Given a domain R, a prime ideal P of R and an upper $\mathscr{P}$ to P, we study in this section the saturated chains of prime ideals between (0) and $\mathscr{P}$ whose length is less than or equal to height P. We will show that essentially anything can happen. More precisely, we will show:

THEOREM D. *Let n, m be two positive integers such that $n+1 \leqslant m \leqslant 2n+1$. Let $u_1, \ldots, u_s$ be integers such that $2 \leqslant u_1 < \cdots < u_s \leqslant n$. Then, there exists a quasi-local domain R with maximal ideal P such that*:

(i) dimension $R = n$

(ii) dimension $R[X] = m$

(iii) *For every upper $\mathscr{P}$ to P, $\{r \mid r$ is the length of some saturated chain of prime ideals between* (0) *and $\mathscr{P}\} = \{u_1, \ldots, u_s, n+1, \ldots, m\}$. If $m = n+1$, there exists such a domain R that is Noetherian.*

Let us see that our Theorem D will be proved if we can construct a quasi-semi-local ring possessing certain properties.

PROPOSITION 4.1. *Let n, m be two positive integrs such that $n+1 \leqslant m \leqslant 2n+1$. Let $u_1, \ldots, u_s$ be integers such that $2 \leqslant u_1 < \cdots < u_s \leqslant n$. Let A be a quasi-semi-local domain with maximal ideals $N_0, N_1, \ldots, N_s$ and let K be a field such that*:

(a) height $N_0 = n$ *and for every upper $\mathscr{N}_0$ to N_0, $\{r \mid r$ is the length of some saturated chain of prime ideals between* (0) *and $\mathscr{N}_0\} = \{n+1, \ldots, m\}$.*

(b) *For $i \in \{1, \ldots, s\}$,* height $N_i = (u_i - 1)$ *and for every upper $\mathscr{N}_i$ to N_i, $\{r \mid r$ is the length of some saturated chain of prime ideals between* (0) *and $\mathscr{N}_i\} = \{u_i\}$.*

(c) *For $i \in \{0, 1, \ldots, s\}$, there exists a surjective homomorphism $\varepsilon_i \colon A \to K$ whose kernel is equal to N_i.*

Let $R = \{a \in A \mid \varepsilon_0(a) = \varepsilon_1(a) = \cdots = \varepsilon_s(a)\}$.

Then, R is a quasi-local domain with maximal ideal $P = N_0 \cap N_1 \cap \cdots \cap N_s$ that satisfies the conditions (i)–(iii) *of Theorem D. If furthermore A is Noethetian, then so is R.*

Proof. By [2, Theorem A, p. 585], R is a quasi-local domain with maximal ideal $P = N_0 \cap N_1 \cap \cdots \cap N_s$ and A is a finite R-module. Hence $\dim R = \dim A = n$ and $\dim R[X] = \dim A[X]$; furthermore $\dim A[X] = m$ since, by Theorem B, $\dim A[X] = \sup\{\text{height}(Q, X) \mid Q \in \operatorname{Spec} A\} = \text{height}(N_0, X) = m$. Thus conditions (i)–(ii) are satisfied and R is Noetherian if A is Noetherian. Now let us look at condition (iii). Let $\mathscr{P}$ be an upper to P. Suppose that there exists a saturated chain of prime ideals of length r in $R[X]$ between (0) and $\mathscr{P}$. By the Going-up Theorem, there exists an integer $i \in \{0, 1, \ldots, s\}$, an upper $\mathscr{N}_i$ to N_i and a saturated chain of prime ideals of length r in $A[X]$ between (0) and $\mathscr{N}_i$; hence $r \in \{u_1, \ldots, u_s, n+1, \ldots, m\}$. Conversely, let $r \in \{u_1, \ldots, u_s, n+1, \ldots, m\}$; we want to show that there exists a saturated chain of prime ideals of length r in $R[X]$ between (0) and $\mathscr{P}$. If $r \in \{n+1, \ldots, m\}$, this is given by Theorem C since dimension $R = n$ and dimension $R[X] = m$. If $r = u_i$ with $i \in \{1, \ldots, s\}$, consider $\mathscr{N}$, an upper to N_i

that lies over $\mathscr{P}$; such $\mathscr{N}$ exists by the Going-up Theorem since the extension $R[X] \hookrightarrow A[X]$ is integral. Since there are only finitely many prime ideals of A lying over P, there are also only finitely many prime ideals of $A[X]$ lying over $\mathscr{P}$ [8, Theorem 2, p. 707]; furthermore they are incomparable between themselves. Let $f(X) \in \mathscr{N}$ such that $f(X)$ belongs to no other prime ideal of $A[X]$ that lies over $\mathscr{P}$ and such that $f(X) \notin N_i[X]$. Since height $N_i = u_i - 1 = r - 1$, there exists by Corollary A.1 a saturated chain of prime ideals in $A[X]$ of length r, $(0) \subset \mathscr{Q}_1 \subset \cdots \subset \mathscr{Q}_r = \mathscr{N}$, such that $f(X) \in \mathscr{Q}_1$. We claim that the chain $(0) \subset \mathscr{Q}_1 \cap R[X] \subset \cdots \subset \mathscr{Q}_r \cap R[X] = \mathscr{P}$ is saturated. Indeed, $\mathscr{Q}_1$ being an upper to zero, $\mathscr{Q}_1 \cap R[X]$ is also an upper to zero and the part $(0) \subset \mathscr{Q}_1 \cap R[X]$ of the chain is saturated. If the remaining part $\mathscr{Q}_1 \cap R[X] \subset \cdots \subset \mathscr{P}$ was not saturated, we could saturate it to get a chain of length strictly bigger; that bigger chain could be lifted to a chain in $A[X]$ of equal length that would start at $\mathscr{Q}_1$ and would necessarily end at $\mathscr{N}$ because $f(X) \in \mathscr{Q}_1$ and $\mathscr{N}$ is the only prime ideal of $A[X]$ that lies over $\mathscr{P}$ and contains $f(X)$; this is absurd because height $N_i = u_i - 1$ and because all the prime ideals between $\mathscr{Q}_1$ and $\mathscr{N}$ are uppers. Thus $(0) \subset \mathscr{Q}_1 \cap R[X] \subset \cdots \subset \mathscr{Q}_r \cap R[X] = \mathscr{P}$ is a saturated chain of prime ideals of length r in $R[X]$ between (0) and $\mathscr{P}$.

Now, our objective is to construct a quasi-semi-local domain A that satisfies the hypothesis of Proposition 4.1. The idea of the construction for general m is the following.

Step 1. For every $i \in \{0, 1, \ldots, s\}$, construct a quasi-local domain B_i that satisfies the conditions that are required to be satisfied by the localization of A at the prime ideal N_i.

Step 2. Intersect those quasi-local domains B_i together to get a quasi-semi-local domain A that is locally equal to the B_i's.

The most delicate part of Step 1 is the construction of B_0; we shall show tat, essentially, the ring constructed by Seidenberg [12] and presented in details by Gilmer [3, Example 3, p. 573] satisfies the required conditions. For $i \in \{1, \ldots, s\}$, it is easy to construct a quasi-local domain B_i that satisfies the required conditions; indeed, in virtue of Corollary 2.4, any valuation ring of dimension $(u_i - 1)$ could be taken; however, the problem here will be to choose those valuation rings in such a way that intersecting them with B_0 gives the "right" ring. The basic idea that will be used in Step 2 comes from the work of Heinzer in [4].

Naturally, when $m > n + 1$, such a domain A cannot be Noetherian. For the special case of $m = n + 1$, we give a construction which, besides being much simpler, yields a domain A that is Noetherian.

EXAMPLE 4.2. Construction of a ring A that satisfies the hypothesis of Proposition 4.1 for general m.

Let n, m be two positive integars such that $n+1 \leqslant m \leqslant 2n+1$. Let $u_1,\ldots, u_s$ be integers such that $2 \leqslant u_1 < \cdots < u_s \leqslant n$.

Step 1. Write $m = n+1+t$; observe that $0 \leqslant t \leqslant n$. Let $k = k_0(Y_1, Y_2,\ldots)$ where k_0 is a field and $Y_1, Y_2,\ldots$ is an infinite number of indeterminates over k_0. Let $Z_1,\ldots, Z_t$, $T_1,\ldots, T_t$, $T_{t+1},\ldots, T_n$ be indeterminates over k.

(a) Let $D_0 = k$. For $1 \leqslant j \leqslant t$, let $D_j = D_{j-1} + T_j k(Z_1, T_1,\ldots, Z_{j-1}, T_{j-1}, Z_j)[T_j]_{(T_j)}$; $J_j = T_j k(Z_1, T_1,\ldots, Z_{j-1}, T_{j-1}, Z_j)[T_j]_{(T_j)}$ is the only prime ideal of height one of D_j [3, Theorem A, (c), p. 560]; observe that J_j is also the maximal ideal of the valuation ring $V_j = k(Z_1, T_1,\ldots, Z_{j-1}, T_{j-1}, Z_j)[T_j]_{(T_j)}$. For $t+1 \leqslant j \leqslant n$, let $D_j = D_{j-1} + T_j k(Z_1, T_1,\ldots, Z_t, T_t, T_{t+1},\ldots, T_{j-1})[T_j]_{(T_j)}$; $J_j = T_j k(Z_1, Z_1,\ldots, Z_t, T_t, T_{t+1},\ldots, T_{j-1})[T_j]_{(T_j)}$ is the only prime ideal of height one of D_j [3, Theorem A, (c), p. 560]; observe that J_j is also the maximal ideal of the valuation ring $V_j = k(Z_1, T_1,\ldots, Z_t, T_t, T_{t+1},\ldots, T_{j-1})[T_j]_{(T_j)}$. Observe that for any j such that $1 \leqslant j \leqslant n$, the set of the non zero prime ideals of D_j is $\{J_q + J_{q+1} + \cdots + J_j \mid q = 1,\ldots, j\}$ which is linearly ordered by inclusion [3, Theorem A, (c), p. 560]; observe also that $D_j = k + J_1 + \cdots + J_j$ and that this sum is direct.

Let $B_0 = D_n$; B_0 is a quasi-local domain with maximal ideal $I_0 = J_1 + \cdots + J_n$ such that dimension $B_0 = n$ and dimension $B_0[X] = m$ [3, Theorem B, p. 574]. We still want to show that if $\mathscr{I}_0$ is an upper to I_0, then $\{r \mid r$ is the length of some saturated chain of prime ideals in $B_0[X]$ between (0) and $\mathscr{I}_0\} = \{n+1,\ldots, m\}$. For that, we will verify the following stronger claim:

Claim 1. If $P \subset Q$ are two prime ideals of B_0 and if $\mathscr{Q}$ is an upper to Q, then $\{r \mid r$ is the length of some saturated chain of prime ideals between $P[X]$ and $\mathscr{Q}\} = \{\text{height}(Q/P) + 1,\ldots, \text{height}(\mathscr{Q}/P[X])\}$.

We first verify three preliminary claims:

Claim 2. If H is any prime ideal of B_0, then B_0/H is integrally closed.

Proof. If $H = (0)$, then $B_0/H = B_0$ is integrally closed [3, Theorem A, (b), p. 560]. If $H \neq (0)$, there exists $1 \leqslant q \leqslant n$ such that $H = J_q + J_{q+1} + \cdots + J_n$, and we have $B_0/H = (D_{q-1} + J_q + \cdots + J_n)/(J_q + \cdots + J_n) \simeq D_{q-1}$ which is integrally closed [3, Theorem A, (b), p. 560].

Claim 3. Let $1 \leqslant j \leqslant n$. If $d \in D_j \backslash J_j$, then $dJ_j = J_j$.

Proof. Since $D_j = D_{j-1} + J_j$ and since $d \in D_j \backslash J_j$, we have $d = \alpha + \beta$ with $\alpha \in D_{j-1} \backslash \{0\}$ and $\beta \in J_j$. Since the quotient field of D_{j-1} is contained in the valuation ring V_j, we have that α is invertible in V_j; since furthermore β belongs to J_j which is the maximal ideal of V_j, we obtain that $d = \alpha + \beta$ is invertible in V_j and consequently that $dJ_j = J_j$.

Claim 4. If H and P are prime ideals of B_0 such that $P \subset H$ and if $\mathscr{P}$ is an upper to P such that $\operatorname{height}(H[X]/\mathscr{P}) = 1$, then $\operatorname{height}(H/P) = 1$.

Proof. Let R be the localization of B_0/P at the prime ideal H/P, let $\mathscr{P}' = \mathscr{P} \cdot R$ and $H' = HR$. Suppose that height $H' > 1$ and let I be a prime ideal of R such that $I \subset H'$ and height $I = 1$. R is integrally closed by Claim 2; then $\mathscr{P}'$ is generated by its polynomials of minimal degree by Lemma 2.1. Since $\mathscr{P}' \subset H'[X]$ is saturated, we have $\mathscr{P}' \not\subseteq I[X]$ and there exists $g(X) \in \mathscr{P}'$, $g(X)$ of minimal degree, such that $g(X) \notin I[X]$; $\mathscr{P}'$ is the only upper to zero that contains $g(X)$ [7, Theorem 36, p. 25]. Let $\bar{g}(X)$ be the canonical image of $g(X)$ in $(R/I)[X]$; $g(X)$ is contained in only a finite number (possibly none) of uppers to I; indeed that number is clearly $\leqslant$ degree $\bar{g}(X)$. Let $\mathscr{Q}_1,..., \mathscr{Q}_t$ be the uppers of I that contain $g(X)$. For $i \in \{1,..., t\}$, we have $g(X) \in \mathscr{Q}_i \backslash I[X]$; then, by Theorem A and by the fact that $\mathscr{P}'$ is the only upper to zero that contains $g(X)$, we obtain that $g(X) \in \mathscr{P}' \subset \mathscr{Q}_i$; this implies that $\mathscr{Q}_i \not\subseteq H'[X]$ since $\mathscr{P}' \subset H'[X]$ is saturated. Since H' is the only maximal ideal of R, since $\mathscr{Q}_i$ is an upper to I that is not contained in $H'[X]$ and since R/I is integrally losed, there exists $\beta_i(X) \in R[X]$ such that $\mathscr{Q}_i = (I[X], \beta_i(X))$ [10, Corollary 2.11, p. 387]; observe that $\beta_i(X) \notin H'[X]$ since $\mathscr{Q}_i \not\subseteq H'[X]$. Then, there exists integers $u_1,..., u_t$, there exists $c \in R \backslash I$ and there exists $q(X) \in I[X]$ such that $g(X) = c(\beta_1(X))^{u_1} \cdots (\beta_t(X))^{u_t} + q(X)$. Since height $I = 1$, since $q(X) \in I[X]$ and $c \in R \backslash I$, we have $(q(X)/c) \in I[X]$ by claim 3 and $(g(X)/c) = (\beta_1(X))^{u_1} \cdots (\beta_t(X))^{u_t} + (q(X)/c) \in R[X]$; even more, since $g(X) = c(g(X)/c) \in \mathscr{P}'$ with $c \notin \mathscr{P}'$, we have $(g(X)/c) \in \mathscr{P}' \subset H'[X]$; this implies that $(\beta_1(X))^{u_1} \cdots (\beta_t(X))^{u_t} = (g(X)/c) - (q(X)/c) \in H'[X]$, which is absurd since $\beta_i(X) \notin H'[X]$ for every $i = 1,..., t$.

Proof of Claim 1. In virtue of Theorem C, it suffices to show that the length of any saturated chain of prime ideals between $P[X]$ and $\mathscr{Q}$ is bigger than or equal to $\operatorname{height}(Q/P) + 1$. Since the prime ideals of B_0 are linearly ordered, it suffices to show that if $\mathscr{P}_1 \subset \mathscr{P}_2$ are two consecutive prime ideals of $B_0[X]$, then $\mathscr{P}_1 \cap B_0$ and $\mathscr{P}_2 \cap B_0$ are either equal or consecutive. Let $P_1 = \mathscr{P}_1 \cap B_0$ and $P_2 = \mathscr{P}_2 \cap B_0$. If $\mathscr{P}_1 = P_1[X]$, it is clear that P_1 and P_2 are either equal or consecutive. If $\mathscr{P}_1$ is an upper to P_1, consider the consecutive prime ideals $\mathscr{P}_1' = \mathscr{P}_1/P_1[X]$ and $\mathscr{P}_2' = \mathscr{P}_2/P_1[X]$ of $B_0[X]/P_1[X] \simeq (B_0/P_1)[X]$; of course, $\mathscr{P}_1'$ is an upper to zero in B_0/P_1, and B_0/P_1 is integrally closed by Claim 2. If $\mathscr{P}_2$ is an upper to P_2, we obtain by Theorem 2.3 that $\operatorname{height}(P_2/P_1) = 1$, i.e., that $P_1 \subset P_2$ are consecutive. If $\mathscr{P}_2 = P_2[X]$, we obtain that $\operatorname{height}(P_2/P_1) = 1$ by Claim 4. This finishes the proof of Claim 1 and shows that the maximal ideal I_0 of B_0 satisfies the properties of Proposition 4.1(a).

(b) Consider $L = k(Z_1,..., Z_t, T_1,..., T_t, T_{t+1},..., T_n)$ the quotient field of B_0 and remember that $u_1,..., u_s$ are integers such that

$2 \leqslant u_1 < \cdots < u_s \leqslant n$. For $i \in \{1,..., s\}$, let G_i be the group $\mathbb{Z}^{(u_i-1)}$ with the lexicographic order; let $e_1,..., e_{u_i-1}$ be the canonical base of $\mathbb{Z}^{(u_i-1)}$. Consider the valuation

$$w_i: L = k(Z_1,..., Z_t, T_1,..., T_n) \to G_i$$

defined in the following way:

- $w_i(\alpha) = 0$ for every $\alpha \in k$.
- $w_i(Z_1) = w_i(Z_2) = \cdots = w_i(Z_t) = 0$.
- For $1 \leqslant j \leqslant (u_{i-1} - 1)$, $w_i(T_j) = e_{j+1}$.
- $w_i(T_{u_{i-1}}) = e_1$.
- For $(u_{i-1} + 1) \leqslant j \leqslant (u_i - 1)$, $w_i(T_j) = e_j$.
- For $u_i \leqslant j \leqslant n$, $w_i(T_j) = 0$.

Those notations make sense even when $i = 1$ if we make the convention that $u_0 = 1$.

Let B_i be the valuation ring of L corresponding to the valuation w_i and let I_i be its maximal ideal. Of course the dimension of B_i is equal to $(u_i - 1)$ and, by Corollary 2.4, I_i satisfies the properties of Proposition 4.1(b). Notice that B_i dominates the local ring

$$k(Z_1,..., Z_t, T_{u_i}, T_{u_i+1},..., T_n)[T_1,..., T_{u_i-1}]_{(T_1,...,T_{u_i-1})}$$

and has the same residue field, namely, $k(Z_1,..., Z_t, T_{u_i}, T_{u_i+1},..., T_n)$.

Step 2. Let $A = B_0 \cap \cdots \cap B_s$; let $N_0 = I_0 \cap A,..., N_s = I_s \cap A$. We intend to prove the following claim:

Claim 5. A is a quasi-semi-local ring whose maximal ideals are $N_0,..., N_s$; it is such that $A_{N_0} = B_0,..., A_{N_s} = B_s$.

Suppose for one moment that Claim 5 has been proved. Then A is a quasi-semi-local ring that clearly satisfies conditions (a) and (b) of Proposition 4.1. Furthermore, we have $A/N_0 = (A_{N_0}/N_0A_{N_0}) = B_0/I_0 \simeq k$ and for $i \geqslant 1$, $A/N_i = (A_{N_i}/N_iA_{N_i}) = B_i/I_i \simeq k(Z_1,..., Z_t, T_{u_i}, T_{u_i+1},..., T_n)$. Since we have taken $k = k_0(Y_1, Y_2,...)$, we obtain that $A/N_i \simeq A/N_0$ for every $i = 1,..., s$; hence condition (c) of Proposition 4.1 is also satisfied and we are through.

Thus, we are left with proving Claim 5. We first observe that:

Claim 6. (a) $B_0,..., B_s$ are incomparable.

(b) $\{\text{maximal ideals of } A\} \subseteq \{N_0,..., N_s\}$.

Proof. (a) Let $i \in \{1,..., s\}$. We have $(T_n/T_1) \in B_0$ but $(T_n/T_1) \notin B_i$;

hence $B_0 \not\subseteq B_i$. On the other hand, B_i is a valuation ring and its dimension is strictly smaller than the dimension of B_0; hence $B_i \not\subseteq B_0$. Let $j \in \{i+1,..., s\}$. We have $(T_{u_{j-1}}/T_1) \in B_j$ but $(T_{u_{j-1}}/T_1) \notin B_i$; hence $B_j \not\subseteq B_i$. On the other hand, B_i is a valuation ring and its dimension is strictly smaller than the dimension of B_j; hence $B_i \not\subseteq B_j$.

(b) Let $x \in A \backslash (N_0 \cup \cdots \cup N_s)$; then $x \in B_i \backslash I_i$ for every $i = 0,..., s$, hence $x^{-1} \in B_i$ for every $i = 0,..., s$, i.e., $x^{-1} \in A$.

In order to prove Claim 5, and in virtue of Claim 6, we just have to show that $A_{N_0} = B_0,..., A_{N_s} = B_s$. We shall do this in the following inductive way:

For $i \in \{0,..., s\}$, let $A_i = B_0 \cap \cdots \cap B_i$, $M_{i,0} = I_0 \cap A_i,..., M_{i,i} = I_i \cap A_i$. By the argument given in Claim 6(b), we know that {maximal ideals of A_i} $\subseteq \{M_{i,0},..., M_{i,i}\}$.

Claim 5.i. $(A_i)_{M_{i,0}} = B_0,..., (A_i)_{M_{i,i}} = B_i$.

Proof. Claim 5.0 is clearly true. Suppose $1 \leqslant i \leqslant s$, and suppose that Claim 5.(i-1) is true. By definition we have $A_i = A_{i-1} \cap B_i$. The Jacobson radical of A_{i-1} is $M_{i-1,0} \cap \cdots \cap M_{i-1,i-1} = I_0 \cap \cdots \cap I_{i-1}$; it is not contained into B_i; indeed, it is easy to check that $(T_1 T_n/T_{u_{i-1}})$ belongs to $I_0 \cap \cdots \cap I_{i-1}$ but does not belong to B_i. Thus, by [4, Proposition 1.13, p. 111], A_{i-1} is a localization of A_i; since, by the hypothesis of induction, $B_0,..., B_{i-1}$ are themselves localizations of A_{i-1}, they are also localizations of A_i and then, we necessarily have $B_0 = (A_i)_{M_{i,0}},..., B_{i-1} = (A_i)_{M_{i,i-1}}$. It remains to see that $B_i = (A_i)_{M_{i,i}}$. That B_i contains $(A_i)_{M_{i,i}}$ is clear. Before working on the other inclusion, let us observe that since $(T_1 T_n/T_{u_{i-1}})$ belongs to the Jacobson radical of A_{i-1}, the elements $1 + (T_1 T_n/T_{u_{i-1}})$ and $x = 1/(1 + (T_1 T_n/T_{u_{i-1}}))$ are invertible in A_{i-1}; furthermore, we have $w_i(T_1 T_n/T_{u_{i-1}}) = (-1, 1, 0,..., 0) < 0 = w_i(1)$, hence $w_i(1 + (T_1 T_n/T_{u_{i-1}})) = (-1, 1, ..., 0)$ and $w_i(x) = (1, -1, 0,..., 0)$. Now, let $y \in B_i$. We know that B_0 and B_i have the same quotient field and that B_0 is a localization of A_i; then we can conclude that B_i and A_i have the same quotient field. Write $y = a/b$ with $a \in A_i$, $b \in A_i \backslash \{0\}$; multiplying them by T_1 if necessary, we can suppose that a and b belong to the Jacobson radical of A_{i-1}. Since $w_i(x) = (1, -1, 0,..., 0)$, there exists an integer p such that $w_i(x^p) > w_i(b)$. With such a p, we have that $w_i(x^p + b) = w_i(b)$, hence that $b/(x^p + b)$ is invertible in B_i. Furthermore, since x is invertible in A_{i-1} and since b belongs to the Jacobson radical of A_{i-1}, we have that $x^p + b$ is invertible in A_{i-1} and therefore that $(b/(x^p + b)) \in A_{i-1}$. Thus, we have $(b/(x^p + b)) \in A_{i-1} \cap B_i = A_i$ and $(b/(x^p + b)) \notin M_{i,i}$. In a similar way, we have $(a/(x^p + b)) \in B_i$; indeed we have $w_i(x_p + b) = w_i(b) \leqslant w_i(a)$ since $y = (a/b) \in B_i$. Since $a/(x^p + b)$ also clearly belongs to A_{i-1}, we obtain that $(a/(x^p + b) \in A_{i-1} \cap B_i = A_i$ and consequently, that $y = a/b = (a/(x^p + b))/(b/(x^p + b)) \in (A_i)_{M_{i,i}}$.

EXAMPLE 4.3. Construction of a Noetherian ring A that satisfies the hypothesis of Proposition 4.1 for $m = n + 1$.

Let n be a positive integer, let $m = n + 1$. Let $u_1,..., u_s$ be integers such that $2 \leqslant u_1 < \cdots < u_s \leqslant n$. Let $k = k_0(Y_1, Y_2,...)$ where k_0 is a field and $Y_1 Y_2,...$ is an infinite number of indeterminates over k_0. Let $X_{01},..., X_{0n}$; $X_{11},..., X_{1(u_1-1)}$;...; $X_{s1},..., X_{s(u_s-1)}$ be indeterminates over k. Let $A = k[X_{01},..., X_{s(u_s-1)}]_S$ where $S = k[X_{01},..., X_{s(u_s-1)}] \backslash ((X_{01},..., X_{0n}) \cup (X_{11},..., X_{1(u_1-1)}) \cup \cdots \cup (X_{s1},..., X_{s(u_s-1)}))$; let $N_0 = (X_{01},..., X_{0n})A$, $N_1 = (X_{11},..., X_{1(u_1-1)})A,..., N_s = (X_{s1},..., X_{s(u_s-1)})A$. It is clear that A is a Noetherian domain whose maximal ideals are $N_0, N_1,..., N_s$. Furthermore, since A is regular, it satisfies conditions (a) and (b) of Proposition 4.1. Finally, because of the way we choose k, it is easy to see that the residue fields are all isomorphic, i.e., that condition (c) is also satisfied.

Remark 4.4. In virtue of Theorem 2.3, it is clear that a domain R satisfying conditions (i)–(iii) of Theorem D cannot be integrally closed in general. However, one can always construct such a domain R seminormal. Indeed, first it is easy to see that this property is satisfied by the ring A constructed in Example 4.2 (as well as in Example 4.3); second, observing that the unique maximal ideal of the ring R constructed in Proposition 4.1 is equal to the Jacobson radical of A, it is also easy to see that the seminormality property goes down to the ring R.

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